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Stability of delay evolution equations with stochastic perturbations

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Abstract. The investigation of stability for hereditary systems is often related to the construction of Lyapunov functionals. The general method of Lyapunov functionals construction, which was proposed by V.Kolmanovskii and L.Shaikhet, is used here to investigate the stability of stochastic delay evolution equations, in particular, for stochastic partial differential equations. This method had already been successfully used for functional-differential equations, for difference equations with discrete time, and for difference equations with continuous time. It is shown that the stability conditions obtained for stochastic 2D Navier-Stokes model with delays are essentially better than the known ones.

1. Introduction.

1.1. Notations and definitions. First of all, we introduce the framework in which our analysis is going to be carried out. Let U, H, K be real, separable Hilbert spaces such that

$$U \subset H \equiv H^* \subset U^*,$$

where U^* is the dual of U and the injections are continuous and dense. We denote by β the constant satisfying

$$|u| \leq \beta \|u\|, \qquad u \in U. \tag{1.1}$$

In particular, we also assume both U and U^* are uniformly convex.

We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in U, H and U^*, respectively; by $\langle\cdot,\cdot\rangle$ the duality product between U^*, U, and by $(\cdot,\cdot)$ the scalar product in H.

Let $W(t)$ be a Q-valued Wiener process on a certain complete probability space $(\Omega, \mathfrak{F}, P)$ which takes values in the separable Hilbert space K, where $Q \in \mathfrak{L}(K,K)$ is a symmetric nonnegative operator and $\mathbf{E}W(t) = 0$, $\mathbf{Cov}(W(t)) = tQ$.

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Let $(\mathfrak{F}_t)_{t\geq 0}$ be the σ-algebras generated by $\{W(s), 0 \leq s \leq t\}$, then $W(t)$ is a martingale relative to $(\mathfrak{F}_t)_{t\geq 0}$ and we have the following representation of $W(t)$:

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i,$$

where $\{e_i\}_{i\geq 1}$ is an orthonormal set of eigenvectors of Q, $\beta_i(t)$ are mutually independent real Wiener processes with incremental covariance $\lambda_i > 0$, $Qe_i = \lambda_i e_i$ and $Tr\, Q = \sum_{i=1}^{\infty} \lambda_i < \infty$ (Tr denotes the trace of an operator, see, for instance, [9]).

For an operator $G \in \mathfrak{L}(K, H)$, the space of all bounded linear operators from K into H, we denote by $\|G\|_2$ its Hilbert-Schmidt norm, i.e.,

$$\|G\|_2^2 = Tr(GWG^*).$$

Given $h \geq 0$, and $T > 0$, we denote by $I^p(-h, T; U)$, $p > 0$, the space of all U–valued processes $(x(t))_{t\in[-h,T]}$ (we will write $x(t)$ for short) measurable (from $[-h, T] \times \Omega$ into U), and satisfying:

1. $x(t)$ is $\mathfrak{F}_t$-measurable almost surely in t, where we set $\mathfrak{F}_t = \mathfrak{F}_0$ for $t \leq 0$;
2. $\int_{-h}^{T} \mathbf{E}\|x(t)\|^p \, dt < +\infty$.

It is not difficult to check that the space $I^p(-h, T; U)$ is a closed subspace of $L^p(\Omega \times [-h, T], \mathfrak{F} \otimes \mathfrak{B}([-h, T]), dP \otimes dt; U)$, where $\mathfrak{B}([-h, T])$ denotes the Borel σ–algebra on $[-h, T]$. We also write $L^2(\Omega; C(-h, T; H))$ instead of $L^2(\Omega, \mathfrak{F}, dP; C(-h, T; H))$, where $C(-h, T; H)$ denotes the space of all continuous functions from $[-h, T]$ into H.

Let $C_H = C([-h, 0], H)$ be the space of all continuous functions from $[-h, 0]$ into H with sup-norm $\|\psi\|_C = \sup_{-h\leq s\leq 0} |\psi(s)|$, $\psi \in C_H$ (the definition is similar for C_U), $L_U^2 = L^2([-h, 0]; U)$ and $L_H^2 = L^2([-h, 0]; H)$.

Given a stochastic process $u(t) \in I^2(-h, T; U) \cap L^2(\Omega; C(-h, T; H))$, we associate with an $L_U^2 \cap C_H$-valued stochastic process $u_t : \Omega \to L_U^2 \cap C_H$, $t \geq 0$, by setting $u_t(s)(\omega) = u(t + s)(\omega)$, $s \in [-h, 0]$.

The aim of this paper is to analyze the stability properties (by means of constructing suitable Lyapunov functionals) of the following class of nonlinear stochastic partial functional differential equations

$$\begin{aligned} du(t) &= (A(t, u(t)) + f(t, u_t))dt + B(t, u_t)dW(t), \qquad t \in [0, T] \\ u(t) &= \psi(t), \qquad t \in [-h, 0], \end{aligned} \tag{1.2}$$

where, in general, the operators are assumed to be nonlinear. In fact, we are interested in the case in which $A(t, \cdot) : U \to U^*$ is a family of nonlinear monotone and coercitive operators, $f(t, \cdot) : C_U \to U^*$ and $B(t, \cdot) : C_U \to \mathfrak{L}(K, H)$ satisfy sublinear properties.

The analysis of the existence and uniqueness of solutions for this model has already been carried out, for instance, in [1,3], and we will not insist in this point here. However, we will explain now which is the concept of solution to be used in our stability analysis.

For a fixed $T > 0$, given an initial value

$$\psi \in I^2(-h, 0; U) \cap L^2(\Omega; C_H),$$

a (variational) solution of (1.2) is a process $u(t) \in I^2(-h,T;U) \cap L^2(\Omega; C(-h,T;H))$ such that

$$\begin{aligned} u(t) =& \psi(0) + \int_0^t [A(s,u(s)) + f(s,u_s)]\, ds \\ &+ \int_0^t B(s,u_s)\, dW(s)\,, \quad P- \text{ a.s.}, \quad \forall t \in [0,T], \\ u(t) =& \psi(t), \quad P-\text{a.}s., \quad \forall t \in [-h,0], \end{aligned} \tag{1.3}$$

where the first equality is defined in U^*.

From now on, as we will be interested in the long-time behavior of the solutions of (1.2), we will assume that (1.3) possesses solutions for all $T > 0$.

Let us denote by $u(\cdot;\psi)$ the solution of Eq. (1.2) corresponding to the initial condition ψ.

Definition 1.1. The trivial solution of Eq. (1.2) is said to be mean square stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mathbf{E}|u(t;\psi)|^2 < \varepsilon$ for all $t \geq 0$ if $\|\psi\|^2_{C_H} = \sup_{s\in[-h,0]} \mathbf{E}|\psi(s)|^2 < \delta$.

Definition 1.2. The trivial solution of Eq. (1.2) is said to be exponentially mean square stable if it is stable and there exists a positive constant λ such that for any $\psi \in C(-h,0,U)$ there exists C (which may depend on ψ) such that $\mathbf{E}|u(t;\psi)|^2 \leq Ce^{-\lambda t}$ for $t > 0$.

Now, as we will use the Itô formula for the solutions of (1.3), we need to define an associate operator L which is usually called the "generator "of equation (1.3).

To calculate the stochastic differential of the process $\eta(t) = v(t,u(t))$, where $u(t)$ is a solution of the equation (1.3) and the function $v(t,u) : [0,\infty) \times U \to \mathbf{R}_+$ has continuous partial derivatives

$$v'_t(t,u) = \frac{\partial v(t,u)}{\partial t}, \qquad v'_u(t,u) = \frac{\partial v(t,u)}{\partial u}, \qquad v''_{uu}(t,u) = \frac{\partial^2 v(t,u)}{\partial u^2},$$

the Itô formula (see, e.g. [10] for more details) is used

$$d\eta(t) = Lv(t,u(t))dt + < v'_u(t,u(t)), B(t,u_t)dW(t) >,$$

where the generator L is defined in the following way

$$\begin{aligned} Lv(t,u(t)) =& v'_t(t,u(t)) + < v'_u(t,u(t)), A(t,u_t) > \\ &+ \frac{1}{2} Tr[v''_{uu}(t,u(t))B(t,u_t)QB^*(t,u_t)]. \end{aligned}$$

The generator L can be applied also for some functionals $V(t,\varphi) : [0,\infty) \times H \to \mathbf{R}_+$. Suppose that a functional $V(t,\varphi)$ can be represent in the form $V(t,\varphi) = V(t,\varphi(0),\varphi(\theta))$, $\theta < 0$ and for $\varphi = u_t$ (or $\varphi(\theta) = u(t+\theta)$) put

$$\begin{aligned} V_\varphi(t,u) &= V(t,\varphi) = V(t,u,\varphi(\theta)), \\ u &= \varphi(0) = u(t), \qquad \theta < 0. \end{aligned} \tag{1.4}$$

Denote by D the set of the functionals, for which the function $V_\varphi(t,u)$ defined by (1.4) has a continuous derivative with respect to t and two continuous derivatives with respect to u. For functionals from D the generator L of the equation (1.3) has the form

$$\begin{aligned} LV(t,u_t) = & V'_{\varphi t}(t,u(t)) + < V'_{\varphi u}(t,u(t)), A(t,u_t) > \\ & + \frac{1}{2} Tr[V''_{\varphi uu}(t,u(t)) B(t,u_t) Q B^*(t,u_t)]. \end{aligned} \tag{1.5}$$

From Itô's formula it follows, that for functionals from D,

$$\mathbf{E}[V(t,u_t) - V(s,u_s)] = \int_s^t \mathbf{E}LV(\tau,u_\tau)d\tau, \qquad t \geq s. \tag{1.6}$$

1.2. Lyapunov type stability theorem. Let us now prove a theorem which will be crucial in our stability investigation.

Theorem 1.1. *Assume that there exists a functional $V(t,u_t)$ such that the following conditions hold for some positive numbers c_1, c_2 and λ:*

$$\mathbf{E}V(t,u_t) \geq c_1 e^{\lambda t} \mathbf{E}|u(t)|^2, \quad t \geq 0, \tag{1.7}$$

$$\mathbf{E}V(0,u_0) \leq c_2 \|\psi\|^2_{C_H}, \tag{1.8}$$

$$\mathbf{E}LV(t,u_t) \leq 0, \quad t \geq 0. \tag{1.9}$$

Then the trivial solution of Eq. (1.2) is exponentially mean square stable.

Proof. Integrating (1.9) via (1.6) we obtain $\mathbf{E}V(t,u_t) \leq \mathbf{E}V(0,u_0)$. From this and (1.7), (1.8) it follows that

$$c_1 \mathbf{E}|u(t)|^2 \leq e^{-\lambda t} \mathbf{E}V(0,u_0) \leq c_2 \|\psi\|^2_{C_H}.$$

The inequality $c_1 \mathbf{E}|u(t)|^2 \leq c_2 \|\psi\|^2_{C_H}$ means that the trivial solution of Eq. (1.2) is stable. Besides, from the inequality $c_1 \mathbf{E}|u(t)|^2 \leq e^{-\lambda t}\mathbf{E}V(0,u_0)$, it follows that the trivial solution of Eq. (1.2) is exponentially mean square stable. □

Note that Theorem 1.1 implies that the stability investigation of Eq. (1.2) can be reduced to the construction of appropriate Lyapunov functionals. The general method of Lyapunov functionals construction is described in [6,7,11,12]. A formal procedure to construct Lyapunov functionals is described below.

1.3. Procedure of Lyapunov functionals construction. The procedure consists of four steps.

Step 1. To transform Eq. (1.2) into the form

$$dz(t,u_t) = (A_1(t,u(t)) + A_2(t,u_t))dt + (B_1(t,u(t)) + B_2(t,u_t))dW(t), \tag{1.10}$$

where $z(t,\cdot)$, $A_2(t,\cdot)$ and $B_2(t,\cdot)$ are families of nonlinear operators, $z(t,0) = 0$, $A_2(t,0) = 0$, $B_2(t,0) = 0$, operators $A_1(t,\cdot)$ and $B_1(t,\cdot)$, such that $A_1(t,0) = 0$, $B_1(t,0) = 0$, and depend only on t and $u(t)$, but do not depend on the previous values $u(t+s)$, $s < 0$.

Step 2. Assume that the trivial solution of the auxiliary equation without memory

$$dy(t) = A_1(t,y(t))dt + B_1(t,y(t))dW(t), \tag{1.11}$$

is exponentially mean square stable and there exists a Lyapunov function $v(t, y(t))$, which satisfies the conditions of Theorem 1.1.

Step 3. A Lyapunov functional $V(t, u_t)$ for Eq.(1.10) is constructed in the form $V = V_1 + V_2$, where $V_1(t, u_t) = v(t, z(t, u_t))$. Here the argument y of the function $v(t, y)$ is replaced on the functional $z(t, x_t)$ from the left-hand part of Eq. (1.10).

Step 4. Usually, the functional $V_1(t, u_t)$ almost satisfies the conditions of Theorem 1.1. In order to fully satisfy these conditions, it is necessary to calculate $\mathbf{E}LV_1(t, u_t)$ and estimate it. Then, the additional functional $V_2(t, u_t)$ can be chosen in a standard way.

Note that the representation (1.10) is not unique. This fact allows, using different representations of the type of (1.10) or different ways of estimating $\mathbf{E}LV_1(t, u_t)$, to construct different Lyapunov functionals and, as a result, to get different sufficient conditions of exponential mean square stability.

2. Construction of Lyapunov functionals for equations with time-varying delay.

Consider the following stochastic evolution equation

$$\begin{gathered} du(t) = (A(t, u(t)) + F(u(t - h(t))))dt + B(t, u(t - \tau(t)))dW(t), \\ h(t) \in [0, h_0], \qquad \tau(t) \in [0, \tau_0], \qquad h = \max[h_0, \tau_0], \\ u(s) = \psi(s), \qquad s \in [-h, 0]. \end{gathered} \tag{2.1}$$

which is a particular case of Eq. (1.2). Here $A(t, \cdot), F : U \to U^*$ are appropriate partial differential operators (see conditions below), $B(t, \cdot) : U \to H$, $W(t)$ is a Q-Wiener process.

We will apply the method described above to construct Lyapunov functionals for Eq. (2.1), and, as a consequence, to obtain sufficient conditions ensuring the stability of the trivial solution.

We will use two different constructions which will provide different stability regions for the parameters involved in the problem.

2.1. The first way of Lyapunov functionals construction.

First we consider a quite general situation for the operators involved in Eq. (2.1).

Theorem 2.1. *Assume that operators in Eq. (2.1) satisfy the conditions*

$$\begin{gathered} \langle A(t, u), u \rangle \le -\gamma \|u\|^2, \qquad \gamma > 0, \\ F : U \to U^*, \qquad \|F(u)\|_* \le \alpha \|u\|, \qquad u \in U, \\ \|B(t, u)\|_2 \le \sigma \|u\|, \qquad u \in U, \end{gathered} \tag{2.2}$$

and

$$\begin{gathered} h(t) \in [0, h_0], \qquad \dot{h}(t) \le h_1 < 1, \\ \tau(t) \in [0, \tau_0], \qquad \dot{\tau}(t) \le \tau_1 < 1. \end{gathered} \tag{2.3}$$

If

$$\gamma > \frac{\alpha}{\sqrt{1 - h_1}} + \frac{\delta}{1 - \tau_1}, \qquad \delta = \frac{1}{2}\sigma^2, \tag{2.4}$$

then the trivial solution of Eq. (2.1) is exponentially mean square stable.

Proof. Owing to the procedure of Lyapunov functionals construction, let us consider the auxiliary equation without memory of the type of (1.11) as

$$\dot{y}(t) = A(t, y(t)). \tag{2.5}$$

The function $v(t,y)=e^{\lambda t}|y|^2$, $\lambda>0$, is a Lyapunov function for Eq. (2.5), i.e., it satisfies the conditions of Theorem 1.1. Actually, it is easy to see that conditions (1.7), (1.8) hold for the function $v(t,y(t))$. Besides, since $\gamma>0$, there exists $\lambda>0$ such that $2\gamma>\lambda\beta^2$. Using (2.5), (1.1) and (2.2), we obtain

$$\frac{d}{dt}v(t,y(t))=\lambda e^{\lambda t}|y(t)|^2+2e^{\lambda t}\langle A(t,y(t)),y(t)\rangle\leq -e^{\lambda t}(2\gamma-\lambda\beta^2)\|y(t)\|^2\leq 0.$$

According to the procedure, we now construct a Lyapunov functional V for Eq. (2.1) in the form $V=V_1+V_2$, where $V_1(t,u_t)=e^{\lambda t}|u(t)|^2$. For Eq. (2.1) via (1.5) and some $\varepsilon>0$ we obtain

$$\begin{aligned}
LV_1(t,u_t)=&\lambda V_1(t,u_t)+2e^{\lambda t}\left\langle A(t,u(t))+F(u(t-h(t))),u(t)\right\rangle+e^{\lambda t}\|B(t,u(t-\tau(t)))\|_2^2\\
\leq& e^{\lambda t}\left[\lambda|u(t)|^2+2\left(-\gamma\|u(t)\|^2+\alpha\|u(t-h(t))\|\|u(t)\|\right)+\sigma^2\|u(t-\tau(t))\|^2\right]\\
\leq& e^{\lambda t}\left[\lambda\beta^2\|u(t)\|^2-2\gamma\|u(t)\|^2+\alpha\left(\varepsilon\|u(t-h(t))\|^2+\varepsilon^{-1}\|u(t)\|^2\right)\right.\\
&\left.+\sigma^2\|u(t-\tau(t))\|^2\right]\\
=&e^{\lambda t}\left[\left(\lambda\beta^2-2\gamma+\frac{\alpha}{\varepsilon}\right)\|u(t)\|^2+\varepsilon\alpha\|u(t-h(t))\|^2+\sigma^2\|u(t-\tau(t))\|^2\right].
\end{aligned}$$

Set now

$$V_2(t,u_t)=\frac{\varepsilon\alpha}{1-h_1}\int_{t-h(t)}^t e^{\lambda(s+h_0)}\|u(s)\|^2ds+\frac{\sigma^2}{1-\tau_1}\int_{t-\tau(t)}^t e^{\lambda(s+\tau_0)}\|u(s)\|^2ds.$$

Then

$$\begin{aligned}
LV_2(t,u_t)=&\frac{\varepsilon\alpha}{1-h_1}\left(e^{\lambda(t+h_0)}\|u(t)\|^2-(1-\dot h(t))e^{\lambda(t-h(t)+h_0)}\|u(t-h(t))\|^2\right)\\
&+\frac{\sigma^2}{1-\tau_1}\left(e^{\lambda(t+\tau_0)}\|u(t)\|^2-(1-\dot\tau(t))e^{\lambda(t-\tau(t)+\tau_0)}\|u(t-\tau(t))\|^2\right)\\
\leq&\frac{\varepsilon\alpha e^{\lambda t}}{1-h_1}\left(e^{\lambda h_0}\|u(t)\|^2-(1-h_1)e^{\lambda(h_0-h(t))}\|u(t-h(t))\|^2\right)\\
&+\frac{\sigma^2e^{\lambda t}}{1-\tau_1}\left(e^{\lambda\tau_0}\|u(t)\|^2-(1-\tau_1)e^{\lambda(\tau_0-\tau(t))}\|u(t-\tau(t))\|^2\right)\\
\leq& e^{\lambda t}\left[\varepsilon\alpha\left(\frac{e^{\lambda h_0}}{1-h_1}\|u(t)\|^2-\|u(t-h(t))\|^2\right)\right.\\
&\left.+\sigma^2\left(\frac{e^{\lambda\tau_0}}{1-\tau_1}\|u(t)\|^2-\|u(t-\tau(t))\|^2\right)\right].
\end{aligned}$$

Thus, for $V=V_1+V_2$ we have

$$LV(t,u_t)\leq e^{\lambda t}\left[\lambda\beta^2-2\gamma+\alpha\left(\frac{1}{\varepsilon}+\frac{\varepsilon e^{\lambda h_0}}{1-h_1}\right)+\sigma^2\frac{e^{\lambda\tau_0}}{1-\tau_1}\right]\|u(t)\|^2.$$

Rewrite the expression in square brackets as

$$-2\gamma+\alpha\left(\frac{1}{\varepsilon}+\frac{\varepsilon}{1-h_1}\right)+\frac{\sigma^2}{1-\tau_1}+\lambda\beta^2+\varepsilon\alpha\frac{e^{\lambda h_0}-1}{1-h_1}+\sigma^2\frac{e^{\lambda\tau_0}-1}{1-\tau_1}.$$

To minimize this expression in the brackets, choose $\varepsilon = \sqrt{1-h_1}$. As a consequence we obtain

$$LV(t,u_t) \le -e^{\lambda t}\left[2\left(\gamma - \frac{\alpha}{\sqrt{1-h_1}} - \frac{\delta}{1-\tau_1}\right) - \rho(\lambda)\right]\|u(t)\|^2 \tag{2.6}$$

with

$$\rho(\lambda) = \lambda\beta^2 + \alpha\frac{e^{\lambda h_0}-1}{\sqrt{1-h_1}} + \sigma^2\frac{e^{\lambda\tau_0}-1}{1-\tau_1}.$$

Since $\rho(0)=0$, then by condition (2.4) there exists $\lambda>0$ small enough such that

$$2\left(\gamma - \frac{\alpha}{\sqrt{1-h_1}} - \frac{\delta}{1-\tau_1}\right) \ge \rho(\lambda).$$

From here and (2.6) it follows that $\mathbf{E}LV(t,u_t) \le 0$. So, the functional $V(t,u_t)$ constructed above satisfies the conditions in Theorem 1.1. This means that the trivial solution of Eq. (2.1) is exponentially mean square stable. $\square$

Note, in particular, if $h(t)\equiv h_0$, $\tau(t)\equiv\tau_0$ then $h_1=0$, $\tau_1=0$ and condition (2.4) takes the form $\gamma > \alpha+\delta$.

2.2. The second way of Lyapunov functionals construction. We now establish a second result which implies that the operator F must be less general than in Theorem 2.1. However, as we will show later in the applications section, the stability regions provided by this theorem will be better than the ones given by Theorem 2.1.

Theorem 2.2. *Suppose that operators in Eq. (2.1) satisfy the following conditions*

$$\begin{gathered}\langle A(t,u)+F(u),u\rangle \le -\gamma\|u\|^2, \qquad \gamma>0,\\ \|A(t,u)+F(u)\|_* \le \alpha_1\|u\|,\\ F:U\to U, \qquad \|F(u)\|_* \le \alpha_2\|u\|, \qquad u\in U,\\ \|B(t,u)\|_2 \le \sigma\|u\|, \qquad u\in U,\end{gathered} \tag{2.7}$$

and

$$\begin{gathered}h(t)\in[0,h_0], \qquad \dot h(t)\le h_1<1, \qquad |\dot h(t)|\le h_2,\\ \tau(t)\in[0,\tau_0], \qquad \dot\tau(t)\le\tau_1<1.\end{gathered} \tag{2.8}$$

If

$$\gamma > \alpha_1\alpha_2h_0 + (1+\alpha_2h_0)\frac{\alpha_2h_2}{\sqrt{1-h_1}} + \frac{\delta}{1-\tau_1}, \tag{2.9}$$

then the trivial solution of Eq. (2.1) is exponentially mean square stable.

Proof. To use the procedure of Lyapunov functionals construction, let us first transform Eq. (2.1) as

$$\begin{aligned}dz(t,u_t) =& (A(t,u(t)) + F(u(t)) + \dot h(t)F(u(t-h(t))))dt\\ &+ B(t,u(t-\tau(t)))dW(t),\end{aligned} \tag{2.10}$$

where

$$z(t,u_t)=u(t)+\int_{t-h(t)}^{t}F(u(s))ds. \tag{2.11}$$

Consider the following auxiliary equation without memory, which is of the type of (1.11), and is given in the form

$$\dot{y}(t)=A(t,y(t))+F(y(t)). \tag{2.12}$$

The function $v(t,y)=e^{\lambda t}|y|^2$ is a Lyapunov function for Eq. (2.12). Actually, since $\gamma>0$ then there exists $\lambda>0$ such that $2\gamma>\lambda\beta^2$. Using (2.12), (1.1), (2.7), we obtain

$$\begin{aligned}\frac{d}{dt}v(t,y(t))=&\lambda e^{\lambda t}|y(t)|^2+2e^{\lambda t}\langle A(t,y(t))+F(y(t)),y(t)\rangle\\ \le&-e^{\lambda t}(2\gamma-\lambda\beta^2)\|y(t)\|^2.\end{aligned}$$

Next, we construct a Lyapunov functional V for Eq. (2.10), (2.11) in the form $V=V_1+V_2$, where

$$V_1(t,u_t)=e^{\lambda t}|z(t,u_t)|^2, \tag{2.13}$$

and $z(t,u_t)$ is defined by (2.11). Using (2.7) for Eq. (2.10), (2.11) and some positive ε_i, $i=1,2,3$, we have

$$\begin{aligned}LV_1(t,u_t)=&\lambda V_1(t,u_t)+2e^{\lambda t}\left\langle A(t,u(t))+F(u(t))+\dot{h}(t)F(u(t-h(t))),z(t,u_t)\right\rangle\\ &+e^{\lambda t}\|B(t,u(t-\tau(t)))\|_2^2\\ =&\lambda V_1(t,u_t)+2e^{\lambda t}\left\langle A(t,u(t))+F(u(t))+\dot{h}(t)F(u(t-h(t))),u(t)\right.\\ &\left.+\int_{t-h(t)}^{t}F(u(s))ds\right\rangle+e^{\lambda t}\|B(t,u(t-\tau(t)))\|_2^2\\ =&\lambda V_1(t,u_t)+2e^{\lambda t}\left\langle A(t,u(t))+F(u(t)),u(t)+\int_{t-h(t)}^{t}F(u(s))ds\right\rangle\\ &+2e^{\lambda t}\dot{h}(t)\left(F(u(t-h(t))),u(t)+\int_{t-h(t)}^{t}F(u(s))ds\right)\\ &+e^{\lambda t}\|B(u(t-\tau(t)))\|_2^2\end{aligned}$$

$$
\begin{aligned}
&\le \lambda V_1(t,u_t) + 2e^{\lambda t}\left[-\gamma\|u(t)\|^2 + \alpha_1\alpha_2\int_{t-h(t)}^t \|u(t)\|\|u(s)\|ds\right] \\
&\quad + 2e^{\lambda t}|\dot{h}(t)|\left(\alpha_2\|u(t-h(t))\|\|u(t)\| + \alpha_2^2\int_{t-h(t)}^t \|u(t-h(t))\|\|u(s)\|ds\right) \\
&\quad + e^{\lambda t}\sigma^2\|u(t-\tau(t))\|^2 \\
&\le \lambda V_1(t,u_t) + e^{\lambda t}\left[-2\gamma\|u(t)\|^2 + \alpha_1\alpha_2\int_{t-h(t)}^t\left(\frac{1}{\varepsilon_1}\|u(t)\|^2 + \varepsilon_1\|u(s)\|^2\right)ds\right] \\
&\quad + e^{\lambda t}|\dot{h}(t)|\left[\alpha_2\left(\varepsilon_2\|u(t-h(t))\|^2 + \frac{1}{\varepsilon_2}\|u(t)\|^2\right)\right. \\
&\quad \left. + \ \alpha_2^2\int_{t-h(t)}^t\left(\varepsilon_3\|u(t-h(t))\|^2 + \frac{1}{\varepsilon_3}\|u(s)\|^2\right)ds\right] + e^{\lambda t}\sigma^2\|u(t-\tau(t))\|^2 \\
&= \lambda V_1(t,u_t) + e^{\lambda t}\sigma^2\|u(t-\tau(t))\|^2 \\
&\quad + e^{\lambda t}\left[\left(-2\gamma + \frac{1}{\epsilon_1}\alpha_1\alpha_2 h(t) + \frac{1}{\varepsilon_2}\alpha_2|\dot{h}(t)|\right)\|u(t)\|^2\right. \\
&\quad + \alpha_2(\varepsilon_2 + \varepsilon_3\alpha_2 h(t))|\dot{h}(t)|\|u(t-h(t))\|^2 \\
&\quad \left. + \alpha_2\left(\varepsilon_1\alpha_1 + \frac{1}{\varepsilon_3}\alpha_2|\dot{h}(t)|\right)\int_{t-h(t)}^t\|u(s)\|^2ds\right].
\end{aligned}
$$

From (2.13) and (2.11) for some $\varepsilon_4 > 0$ it follows that

$$
\begin{aligned}
e^{-\lambda t}V_1(t,u_t) &= |u(t)|^2 + 2\int_{t-h(t)}^t (u(t),F(u(s)))ds + \left|\int_{t-h(t)}^t F(u(s))ds\right|^2 \\
&\le |u(t)|^2 + 2\int_{t-h(t)}^t |u(t)||F(u(s))|ds + h(t)\int_{t-h(t)}^t |F(u(s))|^2ds \\
&\le |u(t)|^2 + \alpha_2\beta^2\int_{t-h(t)}^t\left(\varepsilon_4\|u(t)\|^2 + \frac{1}{\varepsilon_4}\|u(s)\|^2\right)ds \\
&\quad + \alpha_2^2 h(t)\beta^2\int_{t-h(t)}^t\|u(s)\|^2ds \\
&\le (1+\varepsilon_4\alpha_2 h(t))\beta^2\|u(t)\|^2 + \alpha_2\beta^2\left(\frac{1}{\varepsilon_4} + \alpha_2 h(t)\right)\int_{t-h(t)}^t\|u(s)\|^2ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
LV_1(t,u_t) \le\, & e^{\lambda t}\left[\lambda\beta^2(1+\varepsilon_4\alpha_2 h(t)) - 2\gamma + \frac{1}{\epsilon_1}\alpha_1\alpha_2 h(t) + \frac{1}{\varepsilon_2}\alpha_2|\dot{h}(t)|\right]\|u(t)\|^2 \\
& + e^{\lambda t}\alpha_2(\varepsilon_2 + \varepsilon_3\alpha_2 h(t))|\dot{h}(t)|\|u(t-h(t))\|^2 + e^{\lambda t}\sigma^2\|u(t-\tau(t))\|^2 \\
& + e^{\lambda t}\alpha_2\left[\varepsilon_1\alpha_1 + \frac{\alpha_2}{\varepsilon_3}|\dot{h}(t)| + \lambda\beta^2\left(\frac{1}{\varepsilon_4} + \alpha_2 h(t)\right)\right]\int_{t-h(t)}^t\|u(s)\|^2ds
\end{aligned}
$$

$$
\begin{aligned}
\le & e^{\lambda t}\left[\left[\lambda\beta^2(1+\varepsilon_4\alpha_2 h_0)-2\gamma+\frac{1}{\varepsilon_1}\alpha_1\alpha_2 h_0+\frac{1}{\varepsilon_2}\alpha_2 h_2\right]\|u(t)\|^2\right.\\
&+\alpha_2 h_2(\varepsilon_2+\varepsilon_3\alpha_2 h_0)\|u(t-h(t))\|^2+e^{\lambda t}\sigma^2\|u(t-\tau(t))\|^2\\
&\left.+\alpha_2\left[\varepsilon_1\alpha_1+\frac{\alpha_2}{\varepsilon_3}h_2+\lambda\beta^2\left(\frac{1}{\varepsilon_4}+\alpha_2 h_0\right)\right]\int_{t-h_0}^{t}\|u(s)\|^2 ds\right].
\end{aligned}
$$

Put now

$$
\begin{aligned}
V_2(t,u_t)=&\frac{(\varepsilon_2+\varepsilon_3\alpha_2 h_0)\alpha_2 h_2}{1-h_1}\int_{t-h(t)}^{t}e^{\lambda(s+h_0)}\|u(s)\|^2 ds\\
&+\frac{\sigma^2}{1-\tau_1}\int_{t-\tau(t)}^{t}e^{\lambda(s+\tau_0)}\|u(s)\|^2 ds+\alpha_2\left[\varepsilon_1\alpha_1+\frac{\alpha_2}{\varepsilon_3}h_2\right.\\
&\left.+\lambda\beta^2\left(\frac{1}{\varepsilon_4}+\alpha_2 h_0\right)\right]\int_{t-h_0}^{t}e^{\lambda(s+h_0)}(s-t+h_0)\|u(s)\|^2 ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
LV_2(t,u_t)=&\frac{(\varepsilon_2+\varepsilon_3\alpha_2 h_0)\alpha_2 h_2}{1-h_1}\left(e^{\lambda(t+h_0)}\|u(t)\|^2-(1-\dot{h}(t))e^{\lambda(t-h(t)+h_0)}\|u(t-h(t))\|^2\right)\\
&+\frac{\sigma^2}{1-\tau_1}\left(e^{\lambda(t+\tau_0)}\|u(t)\|^2-(1-\dot{\tau}(t))e^{\lambda(t-\tau(t)+\tau_0)}\|u(t-\tau(t))\|^2\right)\\
&+\alpha_2\left[\varepsilon_1\alpha_1+\frac{\alpha_2}{\varepsilon_3}h_2+\lambda\beta^2\left(\frac{1}{\varepsilon_4}+\alpha_2 h_0\right)\right]\\
&\times\left(e^{\lambda(t+h_0)}h_0\|u(t)\|^2-\int_{t-h_0}^{t}e^{\lambda(s+h_0)}\|u(s)\|^2 ds\right)\\
\le & e^{\lambda t}\alpha_2 h_2(\varepsilon_2+\varepsilon_3\alpha_2 h_0)\left(\frac{e^{\lambda h_0}}{1-h_1}\|u(t)\|^2-\|u(t-h(t))\|^2\right)\\
&+e^{\lambda t}\sigma^2\left(\frac{e^{\lambda\tau_0}}{1-\tau_1}\|u(t)\|^2-\|u(t-\tau(t))\|^2\right)\\
&+\alpha_2\left[\varepsilon_1\alpha_1+\frac{\alpha_2}{\varepsilon_3}h_2+\lambda\beta^2\left(\frac{1}{\varepsilon_4}+\alpha_2 h_0\right)\right]\\
&\times\left(e^{\lambda(t+h_0)}h_0\|u(t)\|^2-\int_{t-h_0}^{t}e^{\lambda(s+h_0)}\|u(s)\|^2 ds\right).
\end{aligned}
$$

Since $e^{\lambda t}\le e^{\lambda(s+h_0)}$ for $s\ge t-h_0$ then

$$
\begin{aligned}
LV_2(t,u_t)\le & e^{\lambda t}\left[\alpha_2 h_2(\varepsilon_2+\varepsilon_3\alpha_2 h_0)\left(\frac{e^{\lambda h_0}}{1-h_1}\|u(t)\|^2-\|u(t-h(t))\|^2\right)\right.\\
&+\sigma^2\left(\frac{e^{\lambda\tau_0}}{1-\tau_1}\|u(t)\|^2-\|u(t-\tau(t))\|^2\right)\\
&+\alpha_2\left[\varepsilon_1\alpha_1+\frac{\alpha_2}{\varepsilon_3}h_2+\lambda\beta^2\left(\frac{1}{\varepsilon_4}+\alpha_2 h_0\right)\right]\\
&\left.\times\left(e^{\lambda h_0}h_0\|u(t)\|^2-\int_{t-h_0}^{t}\|u(s)\|^2 ds\right)\right].
\end{aligned}
$$

As a result for $V = V_1 + V_2$ we obtain

$$
\begin{aligned}
LV(t,u_t) \leq & e^{\lambda t}\Bigg(\lambda\beta^2(1+\varepsilon_4\alpha_2 h_0) - 2\gamma + \frac{1}{\varepsilon_1}\alpha_1\alpha_2 h_0 \\
& + \frac{1}{\varepsilon_2}\alpha_2 h_2 + \alpha_2 h_2(\varepsilon_2+\varepsilon_3\alpha_2 h_0)\frac{e^{\lambda h_0}}{1-h_1} + \frac{\sigma^2 e^{\lambda\tau_0}}{1-\tau_1}\Bigg)\|u(t)\|^2 \\
& + e^{\lambda(t+h_0)}\alpha_2 h_0\left[\varepsilon_1\alpha_1 + \frac{1}{\varepsilon_3}\alpha_2 h_2 + \lambda\beta^2\left(\frac{1}{\varepsilon_4}+\alpha_2 h_0\right)\right]\|u(t)\|^2 \\
= & e^{\lambda t}\Bigg[\lambda\beta^2(1+\varepsilon_4\alpha_2 h_0) - 2\gamma + \frac{1}{\varepsilon_1}\alpha_1\alpha_2 h_0 + \frac{1}{\varepsilon_2}\alpha_2 h_2 \\
& + \alpha_2 h_2(\varepsilon_2+\varepsilon_3\alpha_2 h_0)\frac{e^{\lambda h_0}}{1-h_1} + \frac{\sigma^2 e^{\lambda\tau_0}}{1-\tau_1} \\
& + e^{\lambda h_0}\alpha_2 h_0\left[\varepsilon_1\alpha_1 + \frac{1}{\varepsilon_3}\alpha_2 h_2 + \lambda\beta^2\left(\frac{1}{\varepsilon_4}+\alpha_2 h_0\right)\right]\Bigg]\|u(t)\|^2 \\
= & e^{\lambda t}\Bigg[-2\gamma + \alpha_1\alpha_2 h_0\left(\frac{1}{\varepsilon_1}+\varepsilon_1\right) + \alpha_2 h_2\left(\frac{1}{\varepsilon_2}+\frac{\varepsilon_2}{1-h_1}\right) \\
& + \alpha_2^2 h_0 h_2\left(\frac{1}{\varepsilon_3}+\frac{\varepsilon_3}{1-h_1}\right) + \frac{\sigma^2}{1-\tau_1} + \rho_\varepsilon(\lambda)\Bigg]\|u(t)\|^2,
\end{aligned}
\tag{2.14}
$$

where

$$
\begin{aligned}
\rho_\varepsilon(\lambda) = & \lambda\left[\beta^2\left(1+e^{\lambda h_0}\alpha_2^2 h_0^2\right) + \beta^2\alpha_2 h_0\left(\varepsilon_4 + \frac{1}{\varepsilon_4}e^{\lambda h_0}\right)\right] + \frac{\sigma^2(e^{\lambda\tau_0}-1)}{1-\tau_1} \\
& + (e^{\lambda h_0}-1)\alpha_2\left[h_0\left(\varepsilon_1\alpha_1 + \frac{\alpha_2 h_2}{\varepsilon_3}\right) + h_2\frac{(\varepsilon_2+\varepsilon_3\alpha_2 h_0)}{1-h_1}\right].
\end{aligned}
$$

To minimize the right-hand side of inequality (2.14) we put $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon_3 = \sqrt{1-h_1}$ and $\varepsilon_4 = e^{\frac{1}{2}\lambda h_0}$. Then, inequality (2.14) takes the form

$$
\begin{aligned}
LV(t,u_t) \leq & -e^{\lambda t}\Bigg[2\left(\gamma - \alpha_1\alpha_2 h_0 - (1+\alpha_2 h_0)\frac{\alpha_2 h_2}{\sqrt{1-h_1}}\right. \\
& \left. - \frac{\delta}{1-\tau_1}\right) - \rho(\lambda)\Bigg]\|u(t)\|^2,
\end{aligned}
\tag{2.15}
$$

where

$$
\begin{aligned}
\rho(\lambda) = & \lambda\left[\beta^2\left(1+e^{\lambda h_0}\alpha_2^2 h_0^2\right) + 2\beta^2\alpha_2 h_0 e^{\frac{1}{2}\lambda h_0}\right] + \frac{\sigma^2(e^{\lambda\tau_0}-1)}{1-\tau_1} \\
& + (e^{\lambda h_0}-1)\alpha_2\left(\alpha_1 h_0 + \frac{h_2(1+2\alpha_2 h_0)}{\sqrt{1-h_1}}\right).
\end{aligned}
\tag{2.16}
$$

From (2.16) it follows that $\rho(0) = 0$. Thus, there exists $\lambda > 0$ small enough such that from the condition (2.9) we deduce that

$$
2\left(\gamma - \alpha_1\alpha_2 h_0 - (1+\alpha_2 h_0)\frac{\alpha_2 h_2}{\sqrt{1-h_1}} - \frac{\delta}{1-\tau_1}\right) \geq \rho(\lambda). \tag{2.17}
$$

This and (2.15) imply that $\mathbf{E}LV(t,u_t)\le 0$ and, as a consequence, the functional $V(t,u_t)$ constructed above satisfies the conditions (1.8), (1.9). However, we cannot ensure that Theorem 1.1 holds true since the functional $V(t,u_t)$ does not satisfy the condition (1.7). Then, we will proceed in a different way.

From (2.15), (2.17) and (1.6) it follows that there exists $c>0$ such that

$$\mathbf{E}V(t,u_t)-\mathbf{E}V(0,u_0)\le -c\int_0^t e^{\lambda s}\mathbf{E}\|u(s)\|^2ds.$$

Therefore,

$$\int_0^\infty e^{\lambda s}\mathbf{E}\|u(s)\|^2ds\le \frac{1}{c}\mathbf{E}V(0,u_0),\qquad \mathbf{E}V(t,u_t)\le \mathbf{E}V(0,u_0). \tag{2.18}$$

Note also that via (2.11)

$$\begin{aligned}|z(t,u_t)|^2=&\left|u(t)+\int_{t-h(t)}^t F(u(s))ds\right|^2\\ \ge&|u(t)|^2-2\int_{t-h(t)}^t|u(t)||F(u(s))|ds\\ \ge&|u(t)|^2-2\alpha_2\beta\int_{t-h(t)}^t|u(t)|\|u(s)\|ds\\ \ge&|u(t)|^2-\alpha_2\left(|u(t)|^2h(t)+\beta^2\int_{t-h(t)}^t\|u(s)\|^2ds\right)\\ \ge&(1-\alpha_2h_0)|u(t)|^2-\alpha_2\beta^2\int_{t-h_0}^t\|u(s)\|^2ds.\end{aligned} \tag{2.19}$$

From (2.7) it follows that

$$\gamma\|u\|^2\le-\langle A(t,u)+F(u),u\rangle\le\|A(t,u)+F(u)\|_*\|u\|\le\alpha_1\|u\|^2,$$

i.e., $\gamma\le\alpha_1$. Using (2.9) we have $\alpha_2h_0<\gamma\alpha_1^{-1}\le 1$. So, from (2.19) we obtain

$$\mathbf{E}|u(t)|^2\le\frac{\mathbf{E}\left|u(t)+\int_{t-h(t)}^t F(u(s))ds\right|^2+\alpha_2\beta^2\int_{t-h_0}^t\mathbf{E}\|u(s)\|^2ds}{1-\alpha_2h_0}. \tag{2.20}$$

Since

$$\mathbf{E}V(0,u_0)\ge\mathbf{E}V(t,u_t)\ge\mathbf{E}V_1(t,u_t)=e^{\lambda t}\mathbf{E}\left|u(t)+\int_{t-h(t)}^t F(u(s))ds\right|^2,$$

then

$$\mathbf{E}\left|u(t)+\int_{t-h(t)}^t F(u(s))ds\right|^2\le e^{-\lambda t}\mathbf{E}V(0,u_0). \tag{2.21}$$

It is easy to see that there exists $C > 0$ such that $\mathbf{E}V(0, u_0) \leq C\mathbf{E}\|u_0\|^2$. Now, from (2.19)-(2.21) it follows that

$$\mathbf{E}|u(t)|^2 \leq C_0\|u_0\|^2, \qquad C_0 = \frac{C + \alpha_2\left(h_0 + \frac{C}{c}\right)}{1 - \alpha_2 h_0}.$$

Therefore, the trivial solution of Eq. (2.1) is mean square stable.

Thanks to (2.18) we have that there exists $C_1 > 0$ such that

$$e^{\lambda(t-h_0)}\int_{t-h_0}^{t}\mathbf{E}\|u(s)\|^2 ds \leq \int_{t-h_0}^{t} e^{\lambda s}\mathbf{E}\|u(s)\|^2 ds \leq \int_{-h_0}^{\infty} e^{\lambda s}\mathbf{E}\|u(s)\|^2 ds \leq C_1.$$

Hence,

$$\int_{t-h_0}^{t}\mathbf{E}\|u(s)\|^2 ds \leq C_1 e^{\lambda h_0} e^{-\lambda t} \tag{2.22}$$

and from (2.20)-(2.22) it follows that, by conditions (2.7)-(2.9), the trivial solution of Eq. (2.1) is exponentially mean square stable. □

Note that if, in particular, $h(t) = h_0$, then $h_2 = 0$ and condition (2.9) takes the form $\gamma > \alpha_1\alpha_2 h_0 + \frac{\delta}{1-\tau_1}$.

3. Some applications In this section we will show some interesting applications to illustrate how our results work.

3.1. Application to a stochastic 2D Navier-Stokes model. We first consider a stochastic 2D Navier-Stokes model with delay. The deterministic version of this problem has already been analyzed in details in [4]. The stochastic situation has also been considered in [5,14] when the delay function is the same in the diffusion and driving terms. We will analyze the case of different delays in both terms.

Let $\Omega \subset \mathbf{R}^2$ be an open and bounded set with regular boundary Γ, $T > 0$ given, and consider the following functional Navier-Stokes problem:

$$\begin{aligned} du + &\left(-\nu\Delta u + \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i}\right) dt \\ &= \left(-\nabla p + g(t, u_t)\right) dt + \Phi(t, u_t) dW(t) \quad \text{in} \quad (0, T) \times \Omega, \\ &div\ u = 0 \quad \text{in} \quad (0, T) \times \Omega, \\ &u = 0 \quad \text{on} \quad (0, T) \times \Gamma, \\ &u(0, x) = u_0(x), \quad x \in \Omega, \\ &u(t, x) = \psi(t, x), \quad t \in (-h, 0), \quad x \in \Omega, \end{aligned} \tag{3.1}$$

where we assume that $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u_0 the initial velocity field, $g(t, u_t)$ is an external force containing some hereditary characteristic, $\Phi(t, u_t)dW(t)$ represent a stochastic term, where $W(t)$ is the standard Wiener process as we considered in the previous sections, and ψ the initial datum in the interval of time $(-h, 0)$, where h is a positive fixed number.

To begin with we consider the following usual abstract spaces

$$\mathcal{U} = \left\{ u \in (C_0^\infty(\Omega))^2 : div\ u\ = 0 \right\},$$

$H =$ the closure of $\mathcal{U}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u,v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) dx,$$

$U =$ the closure of $\mathcal{U}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$, and associated scalar product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u,v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that $U \subset H \equiv H^* \subset U^*$, where the injections are dense and compact. Now we denote $a(u,v) = ((u,v))$, and define the trilinear form b on $U \times U \times U$ by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall\, u,v,w \in U.$$

Assume that the delay terms are given by

$$g(t,u_t) = Gu(t-h(t)), \qquad \Phi(t,u_t) = \hat{\Phi} u(t-\tau(t)),$$

where $G \in \mathfrak{L}(U,U^*)$ is self-adjoint and $\hat{\Phi} \in \mathfrak{L}(U,H)$, the delay functions $h(t)$ and $\tau(t)$ satisfy the assumptions in Theorem 2.1. Then problem (3.1) can be set in the abstract formulation:

$$\begin{aligned} \text{To find } u \in & I^2(-h,T;U) \cap L^2(\Omega; L^\infty(0,T;H)) \\ & \text{such that for all } v \in U \\ d(u(t),v) &+ (\nu a(u(t),v) + b(u(t),u(t),v))dt \\ &= (Gu(t-h(t)),v)\, dt + (\Phi(t,u_t) dW(t), v), \\ u(0) = u_0,&\ \ u(t) = \psi(t),\ \ t \in (-h,0), \end{aligned} \tag{3.2}$$

where the equation in (3.2) must be understood in the distributional sense of $\mathfrak{D}'(0,T)$.

Observe that Eq. (3.2) can be rewritten as Eq. (2.1) by denoting $A(t,\cdot), F : U \to U^*$ the operators defined as

$$A(t,u) = -\nu a(u,\cdot) - b(u,u,\cdot), \quad F(u) = Gu, \quad B(t,u) = \hat{\Phi} u, \quad u \in U.$$

In the present situation, i.e., for the operators $G \in \mathfrak{L}(U,U^*)$, $\hat{\Phi} \in \mathfrak{L}(U,H)$ and the functions $g(t,u_t) = Gu(t-h(t))$, $\Phi(t,u_t) = \hat{\Phi} u(t-\tau(t))$ defined above, we have that $\gamma = \nu$, $\alpha = \|G\|_{\mathfrak{L}(U,U^*)}$, $\sigma = \|\hat{\Phi}\|_{\mathfrak{L}(U,H)}$, $\beta = \lambda_1^{-1/2}$ (λ_1 is the first eigenvalue of the Stokes operator) and assumptions in Theorem 2.1 hold assuming that

$$\nu > \frac{\|G\|_{\mathfrak{L}(U,U^*)}}{\sqrt{1-h_1}} + \frac{\|\hat{\Phi}\|^2_{\mathfrak{L}(U,H)}}{2(1-\tau_1)}. \tag{3.3}$$

Remark 3.1. Observe that if $G \in \mathfrak{L}(H,H)$ and $\hat{\Phi} \in \mathfrak{L}(H,H)$ then $G \in \mathfrak{L}(U,U^*)$ and $\hat{\Phi} \in \mathfrak{L}(U,H)$, in addition, we have that

$$\begin{aligned}\|G\|_{\mathfrak{L}(U,U^*)} \leq& \lambda_1^{-1}\|G\|_{\mathfrak{L}(H,H)},\\ \|\hat{\Phi}\|_{\mathfrak{L}(U,H)} \leq& \lambda_1^{-1/2}\|G\|_{\mathfrak{L}(H,H)}.\end{aligned}$$

So, if we assume that

$$\nu\lambda_1 > \frac{\|G\|_{\mathfrak{L}(H,H)}}{\sqrt{1-h_1}} + \frac{\|\hat{\Phi}\|^2_{\mathfrak{L}(H,H)}}{2(1-\tau_1)}. \tag{3.4}$$

it also follows (3.3) and, consequently, we have the exponential stability of the trivial solution.

3.2. Application to some stochastic reaction-diffusion equations. In this subsection we will consider three different reaction-diffusion equations to show how we can obtain different stability regions for the parameters involved in the equation.

Let us then consider the following three problems:

$$du(t,x) = \left(\nu\frac{\partial^2 u(t,x)}{\partial x^2} + \mu\frac{\partial^2 u(t-h(t),x)}{\partial x^2}\right)dt + \sigma u(t-\tau(t),x)dW(t), \tag{3.5}$$

$$du(t,x) = \left(\nu\frac{\partial^2 u(t,x)}{\partial x^2} + \mu\frac{\partial u(t-h(t),x)}{\partial x}\right)dt + \sigma u(t-\tau(t),x)dW(t), \tag{3.6}$$

$$du(t,x) = \left(\nu\frac{\partial^2 u(t,x)}{\partial x^2} + \mu\, u(t-h(t),x)\right)dt + \sigma u(t-\tau(t),x)dW(t) \tag{3.7}$$

with the conditions

$$\begin{gathered} t \geq 0, \qquad x \in [a,b], \qquad u(t,a) = u(t,b) = 0,\\ h(t) \in [0,h_0], \qquad \dot{h}(t) \leq h_1 < 1, \qquad |\dot{h}(t)| \leq h_2,\\ \tau(t) \in [0,\tau_0], \qquad \dot{\tau}(t) \leq \tau_1 < 1.\end{gathered} \tag{3.8}$$

where $\nu > 0$ and μ is an arbitrary constant. Note that in all of these situations we can consider $U = H_0^1([a,b])$ and $H = L^2([a,b])$. The constant β for the injection $U \subset H$ equals $\beta = \lambda_1^{-1/2}$, where $\lambda_1 = \pi^2(b-a)^{-2}$ is the first eigenvalue of the operator $-\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions. We can therefore apply Theorem 2.1 to all these examples yielding the following sufficient stability conditions.

For equation (3.5)

$$\nu > \frac{|\mu|}{\sqrt{1-h_1}} + \frac{\sigma^2}{2\lambda_1(1-\tau_1)},$$

for equation (3.6)

$$\nu > \frac{|\mu|}{\sqrt{\lambda_1(1-h_1)}} + \frac{\sigma^2}{2\lambda_1(1-\tau_1)},$$

for equation (3.7)

$$\nu > \frac{|\mu|}{\lambda_1\sqrt{1-h_1}} + \frac{\sigma^2}{2\lambda_1(1-\tau_1)}. \tag{3.9}$$

Note that in the particular case $[a,b] = [0,\pi]$ it holds $\lambda_1 = 1$ and these three conditions given by Theorem 2.1 are the same.

Observe that Theorem 2.2 can be applied only to Eq. (3.7). For this equation the parameters of Theorem 2.2 are $\gamma = \alpha_1 = \nu - \mu\lambda_1^{-1}$, $\alpha_2 = |\mu|\lambda_1^{-1/2}$. It gives the following sufficient stability condition:

$$\begin{aligned}\nu > &\frac{\mu}{\lambda_1} + \frac{|\mu|h_2}{\sqrt{\lambda_1(1-h_1)}}\left(\frac{\sqrt{\lambda_1}+|\mu|h_0}{\sqrt{\lambda_1}-|\mu|h_0}\right)\\ &+ \frac{\sigma^2}{2\sqrt{\lambda_1}(\sqrt{\lambda_1}-|\mu|h_0)(1-\tau_1)}, \qquad |\mu| < \frac{\sqrt{\lambda_1}}{h_0}.\end{aligned} \tag{3.10}$$

Note that the stability condition (3.9) that we have obtained for equation (3.7) improves the one in the paper [2]. Indeed, in the case $[a,b] = [0,\pi]$ and constant delay, i.e., $h(t) = \tau(t) = h$, the stability condition obtained in [2] for $\nu = 1$ is

$$1 > 3e^h(\mu^2 + \sigma^2). \tag{3.11}$$

Also, our stability condition (3.9) improves the one in [8] since in this paper the stability condition is obtained in the form

$$1 > 3(\mu^2 + \sigma^2), \tag{3.12}$$

although the delay functions $h(\cdot)$ and $\tau(\cdot)$ are only assumed to be measurable.

Fig. 3.1 shows the stability regions for the equation (3.7) which have been obtained for the values of parameters $\nu = 1$, $h_1 = \tau_1 = 0$, $\lambda_1 = 1$. The line (1) represents the condition (3.9), the line (2) represents (3.10), (3) corresponds to (3.11), and (4) to (3.12). One can see that both conditions (3.9) and (3.10) are essentially better than the condition (3.12) which is also better than (3.11). On the other hand, the condition (3.10) is worse than (3.9) for $\mu > 0$, but better than (3.9) for $\mu < 0$.

In Fig. 3.2 and 3.3 one can see that the conditions (3.9) and (3.10) complement each other for $\nu > 1$ and $\nu < 1$ (in Fig. 3.2 we have $\nu = 1.2$, while $\nu = 0.7$ in Fig. 3.3) with the same values of other parameters.

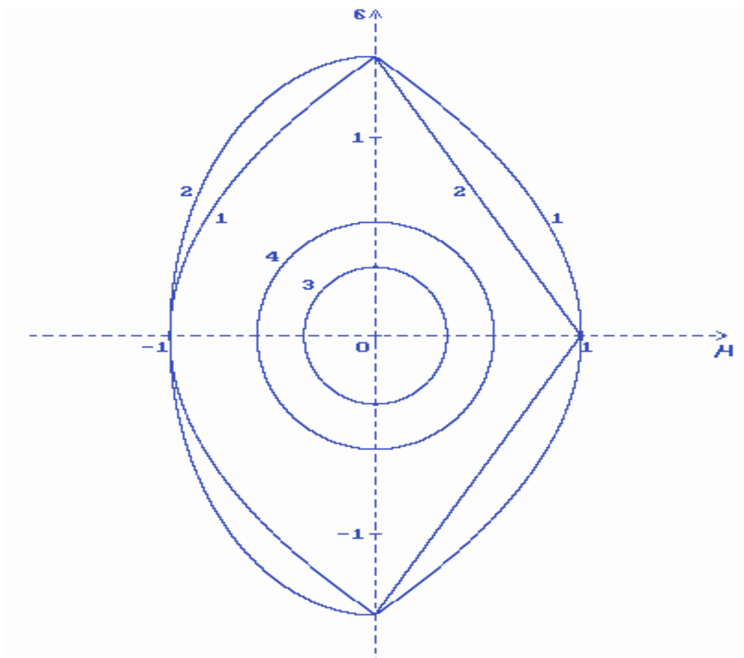

Fig. 3.1.

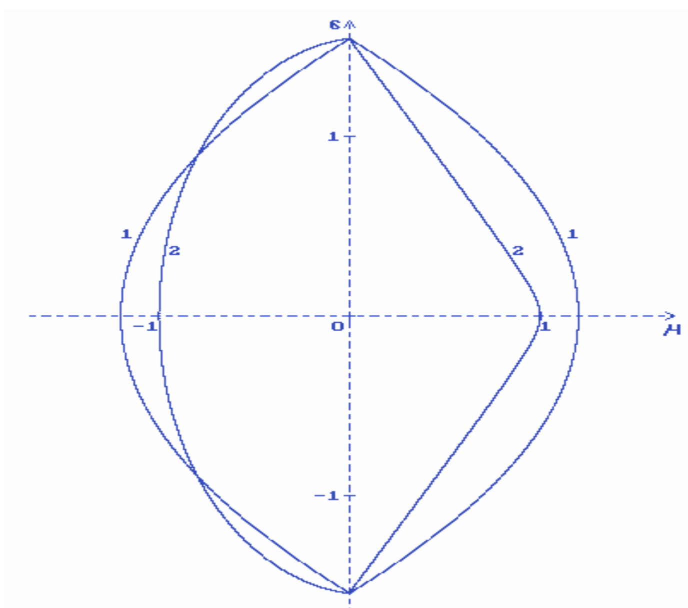

Fig. 3.2.

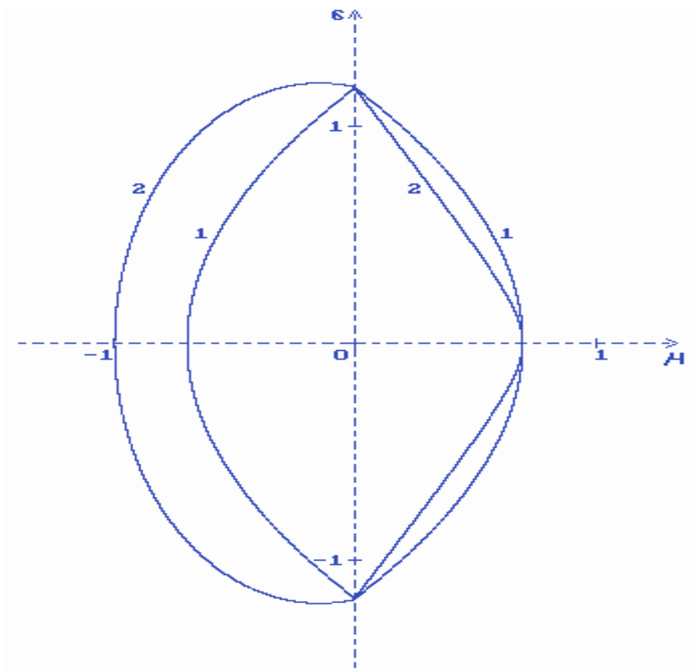

Fig. 3.3.

Let us then consider the following three problems:

$$du(t,x) = \left(\nu \frac{\partial^2 u(t,x)}{\partial x^2} + \mu \frac{\partial^2 u(t-h(t),x)}{\partial x^2}\right) dt + \sigma \frac{\partial u(t-\tau(t),x)}{\partial x} dW(t), \quad (3.13)$$

$$du(t,x) = \left(\nu \frac{\partial^2 u(t,x)}{\partial x^2} + \mu \frac{\partial u(t-h(t),x)}{\partial x}\right) dt + \sigma \frac{\partial u(t-\tau(t),x)}{\partial x} dW(t), \quad (3.14)$$

$$du(t,x) = \left(\nu \frac{\partial^2 u(t,x)}{\partial x^2} + \mu\, u(t-h(t),x)\right) dt + \sigma \frac{\partial u(t-\tau(t),x)}{\partial x} dW(t) \quad (3.15)$$

with the same conditions (3.8), where $\nu > 0$ and μ is an arbitrary constant. We can again apply Theorem 2.1 to all these examples yielding the following sufficient stability conditions.

For equation (3.13)

$$\nu > \frac{|\mu|}{\sqrt{1-h_1}} + \frac{\sigma^2}{2(1-\tau_1)},$$

for equation (3.14)

$$\nu > \frac{|\mu|}{\sqrt{\lambda_1(1-h_1)}} + \frac{\sigma^2}{2(1-\tau_1)},$$

for equation (3.15)

$$\nu > \frac{|\mu|}{\lambda_1\sqrt{1-h_1}} + \frac{\sigma^2}{2(1-\tau_1)}.$$

Note that in the particular case $[a,b]=[0,\pi]$ as $\lambda_1=1$, and these three conditions given by Theorem 2.1 are the same.

Observe that Theorem 2.2 can be applied only to Eq. (3.15). For this equation the parameters of Theorem 2.2 are $\gamma=\alpha_1=\nu-\mu\lambda_1^{-1}$, $\alpha_2=|\mu|\lambda_1^{-1/2}$. It gives the following sufficient stability condition:

$$\begin{aligned}\nu >&\frac{\mu}{\lambda_1}+\frac{|\mu|h_2}{\sqrt{\lambda_1(1-h_1)}}\left(\frac{\sqrt{\lambda_1}+|\mu|h_0}{\sqrt{\lambda_1}-|\mu|h_0}\right)\\ &+\frac{\sigma^2\sqrt{\lambda_1}}{2(\sqrt{\lambda_1}-|\mu|h_0)(1-\tau_1)},\qquad |\mu|<\frac{\sqrt{\lambda_1}}{h_0}.\end{aligned}$$

Remark 3.2. Analogous examples have been analyzed in [13], and similar conditions to ours have been obtained without assuming that the delay function are continuously differentiable. However, the concept of solution used in [13] is stronger than the one we use in this paper, since their proof relies in an equality stated in Theorem 2.2 (see [13] page 492), which implies that the solution must belong to $C^1(0,T;L^2(\Omega;H))$, while in the usual situation, the solution are proved to belong only to $L^2(\Omega;C(0,T;H))$ (see Definition 2.1 in [13] and our definition of solution in this paper) and, consequently, the technique used in [13] cannot be applied. Moreover, the operator in the diffusion part of the equation in [13] does not allow for first order derivatives while it does in our case (see the preceding examples).

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